Convergence of Peridynamics to Classical Elasticity Theory

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Abstract

The peridynamic model of solid mechanics is a nonlocal theory containing a length scale. It is based on direct interactions between points in a continuum separated from each other by a finite distance. The maximum interaction distance provides a length scale for the material model. This paper addresses the question of whether the peridynamic model for an elastic material reproduces the classical local model as this length scale goes to zero. We show that if the motion, constitutive model, and any nonhomogeneities are sufficiently smooth, then the peridynamic stress tensor converges in this limit to a Piola-Kirchhoff stress tensor that is a function only of the local deformation gradient tensor, as in the classical theory. This limiting Piola-Kirchhoff stress tensor field is differentiable, and its divergence represents the force density due to internal forces. The limiting, or *collapsed,* stress-strain model satisfies the conditions in the classical theory for angular momentum balance, isotropy, objectivity, and hyperelasticity, provided the original peridynamic constitutive model satisfies the appropriate conditions.

1 Introduction

The peridynamic model of solid mechanics [1, 2] has been proposed as a way to model the deformation of bodies in which discontinuities, especially cracks, occur spontaneously. The objective of the model is to replace the classical continuum description, which assumes a smooth deformation, so that the basic equations remain applicable even when singularites appear in the deformation. This is in contrast to the classical approach, in which the inability to evaluate the spatial derivatives on a crack leads to the need for the special techniques of fracture mechanics.

In the peridynamic model, we imagine that any body-point $\mathbf{x}$ in the reference configuration is acted upon by forces due to the deformation of all the body-points $\mathbf{x}'$ within some neighborhood of finite radius δ centered at $\mathbf{x}$. The radius δ is called the *horizon*, and the body-points within this neighborhood of $\mathbf{x}$ in the reference configuration are called the *family* of $\mathbf{x}$. The vector $\mathbf{x}' - \mathbf{x}$ is called a *bond*. The interaction between any $\mathbf{x}'$ and $\mathbf{x}$ is expressed in terms of the *force state* at $\mathbf{x}$ at time t, which is written $\underline{\mathbf{T}}[\mathbf{x}, t]$. The force state is a function that associates with any bond $\mathbf{x}' - \mathbf{x}$ a force density $\underline{\mathbf{T}}[\mathbf{x}, t]\langle \mathbf{x}' - \mathbf{x}\rangle$ (per unit volume squared) acting on $\mathbf{x}$. This force density arises due to the internal forces generated by the deformation of the family of $\mathbf{x}$ relative to $\mathbf{x}$. (The bond on which $\underline{\mathbf{T}}$ operates is written in angle brackets to distinguish it from quantities that the force state itself may depend on, such as $\mathbf{x}$ and t.) The physical interpretation of this type of force density, and its relation to Newtonian mechanics, is discussed in the next section.

The basic equations of the peridynamic model [2] include the equation of motion,

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}, t) = \mathbf{L}_{\mathbf{u}}(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t) \qquad \forall \mathbf{x} \in \mathcal{B}, \quad t \geq 0, \tag{1}$$

$$\mathbf{L}_{\mathbf{u}}(\mathbf{x}, t) = \int_{\mathcal{B}} \left\{ \underline{\mathbf{T}}[\mathbf{x}, t]\langle \mathbf{x}' - \mathbf{x}\rangle - \underline{\mathbf{T}}[\mathbf{x}', t]\langle \mathbf{x} - \mathbf{x}'\rangle \right\} \, dV_{\mathbf{x}'} \tag{2}$$

where $\mathcal{B}$ is the reference configuration of the body, ρ is the density in the reference configuration, $\mathbf{u}$ is the displacement, and $\mathbf{b}$ is the body force density. The term $\mathbf{L}_{\mathbf{u}}(\mathbf{x}, t)$ is a functional of displacement that represents the internal force density (per unit volume) that is exerted on $\mathbf{x}$ by other body-points. Since $\underline{\mathbf{T}}[\mathbf{x}, t]$ depends only on the deformation of the family of $\mathbf{x}$, we assume that

$$|\boldsymbol{\xi}| > \delta \quad \Longrightarrow \quad \underline{\mathbf{T}}\langle \boldsymbol{\xi}\rangle = \mathbf{0}. \tag{3}$$

Let $\mathbf{y}$ denote the motion of $\mathcal{B}$ in the usual sense; thus $\mathbf{y}(\mathbf{x}, t)$ is the deformed position at time t of the body-point $\mathbf{x} \in \mathcal{B}$. To express the dependence of $\underline{\mathbf{T}}[\mathbf{x}, t]$ on the deformation of the family of $\mathbf{x}$, we define the *deformation state* $\underline{\mathbf{Y}}[\mathbf{x}, t]$. This is a function that associates with any bond $\mathbf{x}' - \mathbf{x}$ the deformed image of the bond:

$$
\begin{aligned}
\underline{\mathbf{Y}}[\mathbf{x}, t]\langle\mathbf{x}' - \mathbf{x}\rangle &= \mathbf{y}(\mathbf{x}', t) - \mathbf{y}(\mathbf{x}, t) \\
&= (\mathbf{x}' + \mathbf{u}(\mathbf{x}', t)) - (\mathbf{x} + \mathbf{u}(\mathbf{x}, t)).
\end{aligned} \tag{4}
$$

The deformation state contains, in one package, the deformed images of all the bonds in the family, which are in general infinite in number. Similarly, the force state contains, in one package, the forces in all of these bonds. In the peridynamic theory, a constitutive model is provided by a relation between the deformation state and the force state. For a simple, homogeneous body, this relation is expressed in the form

$$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}).$$

The essential idea of peridynamic constitutive modeling is that the force transmitted through a given bond $\mathbf{x}' - \mathbf{x}$ depends on the totality of the deformations of all the bonds as a whole, not just on the deformation of the particular bond $\mathbf{x}' - \mathbf{x}$. (The specialization of the model to the case where each bond responds independently of the others is called *bond-based* peridynamics; this was the original version of the theory proposed in [1].)

As mathematical objects, the force state $\underline{\mathbf{T}}$ and the deformation state $\underline{\mathbf{Y}}$ are examples of *peridynamic vector states*, which are simply functions that map bonds into some other vector quantity. In this sense, peridynamic vector states are similar to second order tensors, but with the important difference that the mapping may be nonlinear or even discontinuous. The set of all vector states is denoted $\mathcal{V}$. Some mathematical properties of states are reviewed in Section 5.

The force state $\underline{\mathbf{T}}[\mathbf{x}, t]$ is analogous to a Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}(\mathbf{x}, t)$, because it contains the totality of all internal forces acting on $\mathbf{x}$ due to the material response at that body-point. Similarly, the deformation state $\underline{\mathbf{Y}}[\mathbf{x}, t]$ is analogous to the deformation gradient tensor $\mathbf{F}(\mathbf{x}, t)$.

An important constitutive model is the *elastic* peridynamic material, defined by

$$\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}) = \nabla \hat{W}(\underline{\mathbf{Y}}) \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V} \tag{5}$$

where $\hat{W} : \mathcal{V} \to \mathbb{R}$ is a differentiable (in the sense of Frechet derivatives) scalar valued function called the *strain energy density function.* The peridynamic equation of motion (1), (2) can be obtained from the Euler-Lagrange equation associated with the following Hamiltonian in an elastic peridynamic body:

$$H_{\mathbf{u}} = \int_0^\infty \int_{\mathcal{B}} \left(\hat{W}(\underline{\mathbf{Y}}[\mathbf{x}, t]) - \frac{\rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}}{2} - \mathbf{b} \cdot \mathbf{u} \right) \, dV_{\mathbf{x}} \, dt \tag{6}$$

Note that the interaction between two continuum body-points $\mathbf{x}$ and $\mathbf{x}'$ in (2) contains contributions from the force states at both $\mathbf{x}$ and $\mathbf{x}'$. These force states are independent of each other from the point of view of constitutive modeling: $\underline{\mathbf{T}}[\mathbf{x}, t]$ depends only on $\underline{\mathbf{Y}}[\mathbf{x}, t]$, while $\underline{\mathbf{T}}[\mathbf{x}', t]$ depends only on $\underline{\mathbf{Y}}[\mathbf{x}', t]$.

The remainder of this paper is organized as follows. Section 2 explains the relation between the force densities in the peridynamic continuum model and forces in the Newtonian mechanics of discrete particles. Section 3 contains a general discussion of the role and significance of forces that act through a finite distance in the peridynamic model and in applications. Section 4 explains this paper's objective of demonstrating that the classical model is recovered from the peridynamic model in the limit of small interaction distance for internal forces, under suitable restrictions. A brief review of mathematical properties of *peridynamic states* is presented in Section 5. These objects provide generalizations of second order tensors that allow all the equations of the peridynamic model, including constitutive models, to be expressed in a concise form. In Section 6 we summarize the properties of the *peridynamic stress tensor*, which explicitly includes nonlocal interactions across finite distances out to the horizon, but nevertheless provides a bridge to the classical idea of the Piola-Kirchhoff stress tensor. Section 7 defines a family of elastic peridynamic materials whose horizon is variable but whose bulk properties are invariant as the horizon is changed. In Section 8 it is shown how this family of peridynamic materials converges in the limit of small horizon to a Piola-Kirchhoff stress tensor that depends on the motion only through the local deformation gradient tensor. It is proved in Section 9 that this converged (or *collapsed*) stress tensor field has the correct properties regarding the relation between its divergence and force density. Properties of the collapsed stress tensor are discussed in Section 10, which shows that the peridynamic conditions for angular momentum balance, isotropy, objectivity, and hyperelasticity carry over to the appropriate classical notions in the limit of zero horizon. Finally, in Section 11, we briefly discuss the implications of these results for jump conditions at interfaces. This section also describes how a given stress tensor from the classical theory can be included within the peridynamic framework; in particular we obtain a force state corresponding to a statistically derived kinetic stress tensor.

2 Particles and peridynamic continua

In this paper, we use the term *body-point* to mean a point $\mathbf{x}$ in the reference configuration of a continuous body that labels a certain bit of matter. A body-point in this sense has zero mass, but a small subregion with volume $V_{\mathbf{x}}$ surrounding $\mathbf{x}$ has a finite mass given by $\rho(\mathbf{x})V_{\mathbf{x}}$, where ρ is the density function. We use the term *discrete particle* to mean a Newtonian particle with finite mass M but zero volume.

These two terms, *body-point* and *discrete particle*, have meanings that are closer to each other within the context of the peridynamic model than within the classical theory. To explain this statement, we illustrate that the peridynamic approach to continuum mechanics consists primarily of a straightforward application of Newton's second law to pairs of small volumes. Let $\mathbf{f}$ denote the *dual force density*, defined by the integrand in the definition of $\mathbf{L_u}$ in (2):

$$\mathbf{f}(\mathbf{x}', \mathbf{x}, t) = \underline{\mathbf{T}}[\mathbf{x}, t]\langle \mathbf{x}' - \mathbf{x} \rangle - \underline{\mathbf{T}}[\mathbf{x}', t]\langle \mathbf{x} - \mathbf{x}' \rangle \tag{7}$$

for any two body-points $\mathbf{x}$ and $\mathbf{x}'$ in $\mathcal{B}$; thus, according to (1),

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}, t) = \int_{\mathcal{B}} \mathbf{f}(\mathbf{x}', \mathbf{x}, t)\; dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x}, t). \tag{8}$$

From (7), observe that

$$\mathbf{f}(\mathbf{x}, \mathbf{x}', t) = -\mathbf{f}(\mathbf{x}', \mathbf{x}, t). \tag{9}$$

Let $\mathbf{p}$ and $\mathbf{q}$ be two body-points in a reference configuration. Suppose that the body consists of two small disjoint regions $\mathcal{B}_\mathbf{p}$ and $\mathcal{B}_\mathbf{q}$ containing $\mathbf{p}$ and $\mathbf{q}$ respectively, thus $\mathcal{B} = \mathcal{B}_\mathbf{p} \cup \mathcal{B}_\mathbf{q}$. Let $V_\mathbf{p} = \text{vol}(\mathcal{B}_\mathbf{p})$ and $V_\mathbf{q} = \text{vol}(\mathcal{B}_\mathbf{q})$. Assume $\mathbf{b} \equiv \mathbf{0}$ and that ρ and $\mathbf{f}$ are bounded and continuous. Integrating both sides of (8) over $\mathcal{B}_\mathbf{p}$,

$$\int_{\mathcal{B}_\mathbf{p}} \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}, t)\; dV_\mathbf{x} = \int_{\mathcal{B}_\mathbf{p}} \left(\int_{\mathcal{B}_\mathbf{p}} + \int_{\mathcal{B}_\mathbf{q}} \right) \mathbf{f}(\mathbf{x}', \mathbf{x}, t)\; dV_{\mathbf{x}'}\; dV_\mathbf{x}. \tag{10}$$

By virtue of (9), we have

$$\int_{\mathcal{B}_\mathbf{p}} \int_{\mathcal{B}_\mathbf{p}} \mathbf{f}(\mathbf{x}', \mathbf{x}, t)\; dV_{\mathbf{x}'}\; dV_\mathbf{x} = \int_{\mathcal{B}_\mathbf{q}} \int_{\mathcal{B}_\mathbf{q}} \mathbf{f}(\mathbf{x}', \mathbf{x}, t)\; dV_{\mathbf{x}'}\; dV_\mathbf{x} = \mathbf{0}, \tag{11}$$

which states that each of the volumes exerts no net force on itself. Omitting terms on the order $o(V_\mathbf{p})$ and $o(V_\mathbf{q})$, equation (10) can therefore be approximated by

$$M_\mathbf{p}\ddot{\mathbf{u}}(\mathbf{p}, t) \approx \mathbf{F_{qp}} \tag{12}$$

where

$$M_\mathbf{p} = \rho(\mathbf{p})V_\mathbf{p}, \qquad \mathbf{F_{qp}} = \mathbf{f}(\mathbf{q}, \mathbf{p}, t)V_\mathbf{q}V_\mathbf{p}. \tag{13}$$

The vector $\mathbf{F_{qp}}$ approximates the total force that $\mathcal{B}_\mathbf{q}$ exerts on $\mathcal{B}_\mathbf{p}$, and $M_\mathbf{p}$ approximates the total mass of $\mathcal{B}_\mathbf{p}$. Similarly,

$$M_\mathbf{q}\ddot{\mathbf{u}}(\mathbf{q}, t) \approx \mathbf{F_{pq}} \tag{14}$$

where

$$M_{\mathbf{q}} = \rho(\mathbf{q})V_{\mathbf{q}}, \qquad \mathbf{F}_{\mathbf{pq}} = \mathbf{f}(\mathbf{p},\mathbf{q},t)V_{\mathbf{p}}V_{\mathbf{q}}. \tag{15}$$

The expressions (12) and (14) are simply approximate statements of Newton's second law for discrete particles at $\mathbf{p}$ and $\mathbf{q}$ containing mass $M_{\mathbf{p}}$ and $M_{\mathbf{q}}$ respectively. The approximations become exact as $\mathcal{B}_{\mathbf{p}}$ and $\mathcal{B}_{\mathbf{q}}$ become infinitesimally small. Thus, in the peridynamic model, small volumes surrounding any two body-points in a body behave like a pair of classical discrete particles. Therefore, in view of (12) and (13), we can interpret $\mathbf{f}(\mathbf{q},\mathbf{p},t)$ as "the force density per unit volume squared that $\mathbf{q}$ exerts on $\mathbf{p}$."

Observe from (9), the second of (13), and the second of (15) that $\mathbf{F}_{\mathbf{pq}} = -\mathbf{F}_{\mathbf{qp}}$, which is consistent with Newton's third law. More generally, it is easy to show that linear momentum is conserved exactly for this system. To see this, reverse the order of integration in the double integral $\int_{\mathcal{B}_{\mathbf{p}}}\int_{\mathcal{B}_{\mathbf{q}}}$ in (10) and use (9) and (11) to show that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{B}_{\mathbf{p}}} \rho(\mathbf{x})\dot{\mathbf{u}}(\mathbf{x},t)\, dV_{\mathbf{x}} &= \int_{\mathcal{B}_{\mathbf{p}}}\int_{\mathcal{B}_{\mathbf{q}}} \mathbf{f}(\mathbf{x}',\mathbf{x},t)\, dV_{\mathbf{x}'}\, dV_{\mathbf{x}} \\
&= -\int_{\mathcal{B}_{\mathbf{p}}}\int_{\mathcal{B}_{\mathbf{q}}} \mathbf{f}(\mathbf{x},\mathbf{x}',t)\, dV_{\mathbf{x}'}\, dV_{\mathbf{x}} \\
&= -\int_{\mathcal{B}_{\mathbf{p}}}\int_{\mathcal{B}_{\mathbf{q}}} \mathbf{f}(\mathbf{x}',\mathbf{x},t)\, dV_{\mathbf{x}}\, dV_{\mathbf{x}'} \\
&= -\int_{\mathcal{B}_{\mathbf{q}}}\int_{\mathcal{B}_{\mathbf{p}}} \mathbf{f}(\mathbf{x}',\mathbf{x},t)\, dV_{\mathbf{x}'}\, dV_{\mathbf{x}} \\
&= -\frac{d}{dt}\int_{\mathcal{B}_{\mathbf{q}}} \rho(\mathbf{x})\dot{\mathbf{u}}(\mathbf{x},t)\, dV_{\mathbf{x}}
\end{aligned} \tag{16}$$

where the change of dummy variables $\mathbf{x} \leftrightarrow \mathbf{x}'$ is used in the third step. (16) establishes that total linear momentum is conserved in $\mathcal{B}$ due to peridynamic interactions.

An example of a dual force density function $\mathbf{f}$ is provided by a distribution of electrostatic charge within a large (nonconductive) body. Let $Q(\mathbf{x})$ denote the charge per unit volume in the reference configuration at any body-point $\mathbf{x}$. Then

$$\mathbf{f}(\mathbf{x}',\mathbf{x},t) = \frac{-Q(\mathbf{x}')Q(\mathbf{x})(\mathbf{y}(\mathbf{x}',t)-\mathbf{y}(\mathbf{x},t))}{4\pi\varepsilon_0|\mathbf{y}(\mathbf{x}',t)-\mathbf{y}(\mathbf{x},t)|^3} \tag{17}$$

where ε_0 is the permittivity of free space. In this example, the peridynamic interaction does not have a finite horizon, *i.e.,* $\delta = \infty$. Equivalently, in terms of the peridynamic force state,

$$\underline{\mathbf{T}}[\mathbf{x},t]\langle\boldsymbol{\xi}\rangle = \frac{-Q(\mathbf{x})Q(\mathbf{x}+\boldsymbol{\xi})\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{8\pi\varepsilon_0|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|^3}, \qquad \boldsymbol{\xi} = \mathbf{x}'-\mathbf{x}. \tag{18}$$

A body composed of *discrete* particles (with finite mass but zero volume) can be represented exactly as peridynamic body using generalized functions. For example, suppose a set of discrete particles is given with reference positions $\mathbf{X}_i$ and masses M_i, $i = 1, 2, \ldots, N$. Let the force exerted by particle j on particle i at time t be denoted by $\mathbf{F}_{ji}(t)$. Define a peridynamic body by

$$
\begin{aligned}
\rho(\mathbf{x}) &= \sum_i M_i \delta_{\mathrm{d}}(\mathbf{x} - \mathbf{X}_i) \\
\mathbf{f}(\mathbf{x}', \mathbf{x}, t) &= \sum_i \sum_{j \neq i} \mathbf{F}_{ji}(t) \delta_{\mathrm{d}}(\mathbf{x}' - \mathbf{X}_j) \delta_{\mathrm{d}}(\mathbf{x} - \mathbf{X}_i)
\end{aligned}
\tag{19}
$$

for all $\mathbf{x}$, $\mathbf{x}'$ in $\mathbb{R}^3$, where δ_{d} denotes the Dirac delta function in three dimensions. (Note that the units of δ_{d} are volume^{-1}.) Then the peridynamic equation of motion in the form (8), after carrying out the integration over $\mathbf{x}'$, implies

$$
M_i \ddot{\mathbf{u}}(\mathbf{X}_i, t) = \sum_j \mathbf{F}_{ji}(t), \qquad i = 1, 2, \ldots, N, \tag{20}
$$

which shows that this peridynamic model of a system of interacting discrete particles follows Newton's second law.

Homogenization procedures, as well as representation of the mass density in terms of probability measures, provide ways of converting the set of discrete particles described by (19) into a more conventional continuum (with continuous density) within the peridynamic framework. Simple examples of such methods are given in [3]. This reference also shows how to construct a peridynamic representation of a system of discrete particles in which the particle positions are characterized through probability density functions. In this case, the probability density functions deform along with the continuum. The probabilistic characterization of particle positions allows the resulting peridynamic model to explicitly include the changes in the elastic energy of the system due to random motions. For example, in a Lennard-Jones crystal, the random fluctuations in position couple with nonlinearities in the interatomic potential, increasing the elastic energy over what it would be without the fluctuations. This type of effect is largely responsible for thermal stress in solids.

3 Length scales, nonlocality, and continua

What is the significance of the length scale δ in the peridynamic model, and how should it be chosen? As noted in [1], for any $\delta > 0$, the parameters in a peridynamic constitutive model can be chosen to match the bulk elastic properties of a material. So, if the only requirement for a peridynamic constitutive model is to reproduce the bulk properties, then δ is essentially arbitrary. (This flexibility in choosing δ is of crucial importance in the development starting in Section 7 below.) The same is true of model parameters related to damage: for any δ, these can be chosen to reproduce the energy release rate of a material [4]. Regardless of the exact value of δ, the peridynamic equations retain their applicability on crack tips and surfaces. If it is additionally required to reproduce effects controlled by small-scale behavior, such as dispersion curves, then δ must be chosen to reflect the relevant physical length scale [5, 6, 7]. An example of a physically determined length scale is the spacing of aggregate in concrete; Bazant has shown that a suitable nonlocal model for concrete has advantages over a homogenized local model in the prediction of damage and localization [8]. A peridynamic approach to fracture in concrete is described by Gerstle, Sau, and Silling [9]. If the peridynamic model is used to represent atomic-scale interactions, as in (19), then the relevant δ would be the cut-off distance for the interatomic potential. Techniques are under development for scaling up peridynamic models derived from small-scale physics to use larger values of δ that are more convenient for macroscale modeling [3].

Is it important to include a length scale in a continuum theory? To explore this question, we examine the relevant assumptions in the classical theory and compare them with the physical nature of matter and technological needs. In this paper, the term *separated force* means a force that acts directly between continuum body-points or discrete particles separated from each other by a finite distance. The physical origin of these forces is not of immediate concern; they could be electrostatic, quantum mechanical, or other. With this definition, because of the finiteness of atoms, all forces within a real physical body are evidently separated, although the actual length scale involved in the physical interactions can vary widely.

We use the term *strongly nonlocal* to describe a mathematical continuum model that includes separated forces explicitly. The peridynamic model is strongly nonlocal. The classical model of solid mechanics is, in contrast, *local*. This term is often thought of as meaning that "the stress depends on the deformation only through the first spatial derivatives." However, there are additional aspects of locality that are inconsistent with the nature of forces in real materials.

For example, consider a continuous body $\mathcal{B}$, and imagine a plane $\mathcal{P}$ through its interior that divides the body into open subregions $\mathcal{B}_+$ and $\mathcal{B}_-$. In the classical theory, it is assumed that the force exerted by $\mathcal{B}_-$ on $\mathcal{B}_+ \cup \mathcal{P}$ is applied on the plane $\mathcal{P}$ itself. Therefore, in the classical view, internal forces are assumed to be *contact forces*. Truesdell [10] cites the *cut principle* of Euler and Cauchy as the origin of this assumption.

Is the assumption of contact forces in a continuum a good enough approximation for all purposes? One case in which it is apparently not good enough is when the geometrical scale of the problem is comparable to the length scale of the forces. Such cases arise in modeling small structures, a situation that is becoming quite common in many technological applications. A striking example is the atomic force microscope (AFM), in which mechanical forces between a probe tip and a sample are measured and interpreted to reveal images of individual atoms on the sample surface, with a length scale on the order of 0.4nm [11]. The length scales that characterize these forces in the AFM are on the same order as the geometry of the structures involved. Similar considerations apply to the mechanics of biological materials. For example, the mechanical force between two individual biological molecules can be measured directly as a function of distance between them out to distances exceeding 100nm [12, 13]. Nanoscale structures are also being designed at length scales on the order of interatomic distances, thus requiring the analysis of forces whose interaction distance is of the same magnitude.

Even in problems with larger geometrical length scales, it may be important to treat internal forces within a continuous body as separated forces rather than as contact forces. The reason is that geometrically small features within a material can have an influence over greater distances than their own length scale or the apparent length scale of the forces involved. Maranganti and Sharma [14] have estimated the length scales at which the classical model of elasticity breaks down for some real materials. They report that in many materials, the length scale relevant to forces that determine the bulk properties of materials far exceeds the interatomic spacing. This is particularly true of heterogeneous materials.

Israelachvili [15] demonstrates that the effects of physical forces between bodies on the colloidal length scale drops off much more slowly than would be suggested just by considering the underlying interatomic force. For example, the van der Waals force between two atoms separated by a distance r is roughly proportional to r^{-6}. Yet the van der Waals force between two parallel cylinders varies with $d^{-3/2}$, where d is the separation distance. Thus, the net force between the cylinders drops off much more slowly than underlying interatomic force as the cylinders are moved apart. The closely related phenomenon of surface tension is another example of separated forces having an important effect at the macroscale; it is also an effect that is not easily incorporated into the classical equations of continuum mechanics.

Regardless of the size of a body, a motion may create its own length scale as it evolves. The classic example of this is fracture, in which the global mechanics of the problem are dictated by what happens in a process zone much smaller than the overall geometretic features of the body. It has been demonstrated through molecular dynamics simulations that certain aspects of fracture, including the role of dislocations, can be understood using interatomic forces acting across finite distances [16]. Eringen, Speziale, and Kim [17] showed that nonlocality in a continuum model has a major effect on the predicted crack tip field in an elastic lattice.

It has been shown by Irving and Kirkwood [18], Noll [19], Hardy [20], Murdoch [21] and others that for a system of discrete particles interacting through a central potential, it is possible to define a tensor field $\mathbf{S}$ such that the classical equation of motion

$$\rho\ddot{\mathbf{u}} = \nabla \cdot \mathbf{S} + \mathbf{b} \tag{21}$$

is satisfied, where ρ, $\mathbf{u}$, and $\mathbf{b}$ are statistically defined quantities representing continuum notions of density, displacement, and body force density respectively. However, the fact that (21) holds for such an $\mathbf{S}$ does not justify the assumption of contact forces with a continuous body. On the contrary, it undermines this assumption, because the tensor field $\mathbf{S}$ is found to be strongly nonlocal. It also contradicts the assumption that the internal forces at a body-point depend only on the first spatial derivatives of the motion. At the end of Section 8, we will return to the question of how accurate the classical model is in the presence of separated forces.

4 Objective

From all of the above considerations, one is led to the conclusion that a continuum model that permits a finite length scale for force interactions could potentially be highly relevant to current trends in technology. The peridynamic model treats all forces as separated, thus avoiding the need to assume contact forces. It also avoids the use of the spatial derivatives of the motion, allowing the modeling of motions that are less than classically smooth, including cracks.

The purpose of this paper is to demonstrate the convergence of the peridynamic model to the classical theory of continuum mechanics, and to state the conditions under which this convergence is obtained. In this sense, to the extent that the peridynamic model adequately represents all the relevant separated forces in a real system, these results provide a clarification of the conditions under which the classical theory is applicable. It does this by showing that the equations of the classical theory are recovered in the sense of a limit as the length scale for interactions approaches zero.

The results obtained include the classical equation of motion (a partial differential equation) and a local material model that, for any $\mathbf{x} \in \mathcal{B}$, depends only on the first spatial derivatives of the motion at $\mathbf{x}$. The convergence of the peridynamic model to the classical model does not depend on the assumption central force interactions. The limiting (local) constitutive model is therefore not restricted to a Poisson ratio of 1/4, which is implied by central forces in an isotropic solid.

To demonstrate the convergence of the peridynamic to the classical model, the equation of motion (1) is expressed in terms of the peridynamic stress tensor field resulting in a PDE that is formally identical to the classical equation of motion [23]. The elastic material model is parameterized by the length scale δ in such a way that the bulk properties of the material under homogeneous deformation are independent of δ. *Subject to the assumptions of sufficient smoothness of the motion and of the constitutive model,* it is then shown that in the limit of small δ, the peridynamic stress tensor field approaches a limit $\boldsymbol{\nu}^0$ that is a differentiable function of $\mathbf{x}$, thus supplying an admissible Piola-Kirchhoff stress tensor field in the classical formulation of the equation of motion. This Piola-Kirchhoff stress tensor is a function of the local deformation gradient tensor.

We further show that the functional $\mathbf{L_u}$ approaches $\nabla \cdot \boldsymbol{\nu}^0$, where $\nabla\cdot$ denotes the divergence operator. The Cauchy stress tensor corresponding to $\boldsymbol{\nu}^0$ is symmetric whenever the underlying peridynamic constitutive model $\hat{\underline{\mathbf{T}}}$ satisfies the appropriate condition for balance of angular momentum. Isotropy and objectivity of $\boldsymbol{\nu}^0$ also hold whenever $\hat{\underline{\mathbf{T}}}$ has these properties.

Convergence of the peridynamic equations in the limit of small δ, as well as other important results related to well-posedness and uniqueness, was established by Emmrich and Weckner [22] for the special case of a linear, isotropic material within the *bond-based* version of the peridynamic theory. This version differs from the more general *state-based* theory considered in the present paper. In the bond-based theory, internal forces within a body occur only due to central force interactions along bonds, and each bond responds independently of all the others. One implication of the bond-based theory is that the bulk properties of a linear isotropic microelastic material necessarily have a Poisson ratio of 1/4. The development in [22] relies on the linearity of the problem. In contrast, the present paper takes a more direct approach that exploits the peridynamic stress tensor [23] and is more generally applicable to nonlinear constitutive models and large motions.

5 Peridynamic states

Consider a body-point $\mathbf{x}$ in a peridynamic body $\mathcal{B}$, and let the horizon be $\delta > 0$. Let the *family* of $\mathbf{x}$, denoted $\mathcal{H}$, be the closed neighborhood in $\mathcal{B}$ of radius δ with center $\mathbf{x}$. For any $\mathbf{x}' \in \mathcal{H}$, the vector $\mathbf{x}' - \mathbf{x}$ is called a *bond*.

A *state* of order m is a mapping that associates with each bond $\boldsymbol{\xi} \in \mathcal{H}$ a tensor of order m denoted $\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle$. A state $\underline{\mathbf{A}}$ is defined when $\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle$ is defined for every $\boldsymbol{\xi} \in \mathcal{H}$. A state of order 0 is called a *scalar state*, a state of order 1 is called a *vector state*, and a state of order 2 is called a *tensor state*. The set of all vector states is denoted $\mathcal{V}$. The peridynamic theory uses vector states as the fundamental quantities for purposes of describing the motion and internal forces near a body-point. Therefore, the role of vector states in the peridynamic theory is similar to that of second order tensors in the classical theory.

A number of notational conveniences have been introduced in [2] for manipulating states. Some of the more important notation for present purposes is summarized below. In the following, $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ are vector states, and $\mathbf{C}$ is a second order tensor.

The product $\mathbf{C}\underline{\mathbf{A}}$ is a vector state defined by

$$(\mathbf{C}\underline{\mathbf{A}})\langle\boldsymbol{\xi}\rangle = \mathbf{C}(\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle) \qquad \forall\boldsymbol{\xi} \in \mathcal{H}. \tag{22}$$

The dot product of two vector states is defined by

$$\underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}} \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}} \tag{23}$$

where the symbol $\cdot$ denotes the usual Cartesian scalar product of two vectors in $\mathbb{R}^3$. Expressed in component form, the dot product of two vector states is written as

$$\underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}} \underline{A}_i\langle\boldsymbol{\xi}\rangle \underline{B}_i\langle\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}}$$

where the $\underline{A}_i\langle\boldsymbol{\xi}\rangle$ are the components of $\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle$ in an orthonormal basis, and where the summation convention is in effect. The composition $\underline{\mathbf{A}} \circ \underline{\mathbf{B}}$ of two vector states is a vector state defined by

$$(\underline{\mathbf{A}} \circ \underline{\mathbf{B}})\langle\boldsymbol{\xi}\rangle = \underline{\mathbf{A}}\langle\underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle\rangle \qquad \forall\boldsymbol{\xi} \in \mathcal{H}.$$

A useful quantity is the *identity* vector state $\underline{\mathbf{X}}$ defined by

$$\underline{\mathbf{X}}\langle\boldsymbol{\xi}\rangle = \boldsymbol{\xi} \qquad \forall\boldsymbol{\xi} \in \mathcal{H}. \tag{24}$$

Observe that

$$(\mathbf{C}\underline{\mathbf{X}})\langle\boldsymbol{\xi}\rangle = \mathbf{C}\boldsymbol{\xi} \qquad \forall\boldsymbol{\xi} \in \mathcal{H},$$

therefore $\mathbf{C}\underline{\mathbf{X}}$ is essentially the same linear transformation as $\mathbf{C}$ expressed in the form of a vector state. (But recall that not all vector states represent linear transformations.) Also note that

$$\underline{\mathbf{A}} \circ \underline{\mathbf{X}} = \underline{\mathbf{X}} \circ \underline{\mathbf{A}} = \underline{\mathbf{A}}.$$

Now we review some properties of functions of states (see [2] for more details). Suppose $\Psi(\cdot) : \mathcal{V} \to \mathbb{R}$ is a scalar valued function of a vector state $\underline{\mathbf{A}}$. For any differential $d\underline{\mathbf{A}}$, let $d\Psi$ be defined by

$$d\Psi = \Psi(\underline{\mathbf{A}} + d\underline{\mathbf{A}}) - \Psi(\underline{\mathbf{A}}).$$

If there exists a vector state valued function $\nabla\Psi(\cdot)$ such that

$$d\Psi = \nabla\Psi(\underline{\mathbf{A}}) \bullet d\underline{\mathbf{A}} \tag{25}$$

for any $\underline{\mathbf{A}}$ and $d\underline{\mathbf{A}}$, then $\nabla\Psi$ is called the Frechet derivative of Ψ, and Ψ is said to be differentiable. Geometrically, the Frechet derivative of Ψ can be thought of as the state whose "direction" results in the maximum incremental change in Ψ, thus providing an infinite-dimensional analogue of the familiar directional derivative of a function on $\mathbb{R}^3$.

If $\underline{\mathbf{\Psi}}(\cdot)$ is a state-valued function of a vector state $\underline{\mathbf{A}}$, then its Frechet derivative $\nabla\underline{\mathbf{\Psi}}\langle\cdot,\cdot\rangle$ is a *rank 2* state field, which means simply that it operates on two bonds rather than one. In this case we alter the notation for the dot product so that the Frechet derivative is defined by

$$d\underline{\mathbf{\Psi}}\langle\boldsymbol{\xi}\rangle = (\nabla\underline{\mathbf{\Psi}} \bullet d\underline{\mathbf{A}})\langle\boldsymbol{\xi}\rangle = \int_{\mathcal{H}} \nabla\underline{\mathbf{\Psi}}\langle\boldsymbol{\xi},\boldsymbol{\xi}'\rangle d\underline{\mathbf{A}}\langle\boldsymbol{\xi}'\rangle \; dV_{\boldsymbol{\xi}'} \qquad \forall\boldsymbol{\xi}\in\mathcal{H}.$$

for any differential vector state $d\underline{\mathbf{A}}$. $\nabla\underline{\mathbf{\Psi}}$ has one higher order than $\underline{\mathbf{\Psi}}$ as well as one higher rank. Thus, if $\underline{\mathbf{\Psi}}$ is a vector state valued function, then $\nabla\underline{\mathbf{\Psi}}$ is a tensor state valued function, and we write

$$d\underline{\Psi}_i\langle\boldsymbol{\xi}\rangle = \int_{\mathcal{H}} (\nabla\underline{\mathbf{\Psi}}\langle\boldsymbol{\xi},\boldsymbol{\xi}'\rangle)_{ij} d\underline{A}_j\langle\boldsymbol{\xi}'\rangle \; dV_{\boldsymbol{\xi}'}. \tag{26}$$

The following notational convenience will be adopted in the remainder of this paper: if $\mathbf{a}$ and $\mathbf{b}$ are vectors and $\mathbf{D}$ is a tensor of order 3, then the quantities $\mathbf{Da}$ and $\mathbf{Dab}$ are defined through components in an orthonormal basis according to

$$(\mathbf{Da})_{ij} = D_{ijk}a_k, \qquad (\mathbf{Dab})_i = D_{ijk}a_j b_k.$$

6 Peridynamic stress tensor

Previous results [23] have shown that the peridynamic equation of motion (1) expressed in the form

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{B}} \mathbf{f}(\mathbf{x}',\mathbf{x},t) \; dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x},t), \tag{27}$$

where $\mathbf{f}$ is the dual force density, is equivalent to the following partial differential equation:

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \nabla\cdot\boldsymbol{\nu}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t) \tag{28}$$

where the symbol $\nabla\cdot$ denotes the divergence operator. Here, $\boldsymbol{\nu}$ is the *peridynamic stress tensor field* defined by

$$\boldsymbol{\nu}(\mathbf{x},t) = \frac{1}{2}\int_{\mathcal{S}}\int_0^\delta\int_0^\delta (y+z)^2 \, \mathbf{f}(\mathbf{x}+y\mathbf{m},\mathbf{x}-z\mathbf{m},t)\otimes\mathbf{m} \; dz \; dy \; d\Omega_{\mathbf{m}} \tag{29}$$

where $\otimes$ denotes the dyadic product of two vectors: $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$. $\mathcal{S}$ is the unit sphere, and $d\Omega_{\mathbf{m}}$ is a differential solid angle in the direction of any unit vector $\mathbf{m}$, which is the dummy variable of integration in the outer integral. (As noted in [23], (29) can be obtained from related expressions in [19].) If $\mathbf{f}$ is given by (7), then it satisfies (9). (Recall from (16) that this ensures balance of linear momentum in the peridynamic model.) Combining (7), (9), and (29) allows the peridynamic stress tensor to be written as

$$\boldsymbol{\nu}(\mathbf{x}, t) = \int_{\mathcal{S}} \int_0^{\delta} \int_0^{\delta} (y+z)^2 \, \underline{\mathbf{T}}[\mathbf{x} - z\mathbf{m}, t]\langle (y+z)\mathbf{m} \rangle \otimes \mathbf{m} \, dz \, dy \, d\Omega_{\mathbf{m}}. \quad (30)$$

$\underline{\mathbf{T}}$ appears only once in this expression, rather than twice as in (7), because both terms turn out to be equal after evaluating the triple integral. In the following sections, the peridynamic stress tensor field given by (30), under suitable parameterization of the constitutive model, will be shown to converge to an admissible Piola-Kirchhoff stress tensor field in the classical theory, provided the motion and constitutive model are sufficiently smooth.

7 Parameterization of an elastic peridynamic material model

This paper will be concerned with elastic peridynamic materials as described in [2]:

$$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}) = \nabla \hat{W}(\underline{\mathbf{Y}}) \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V}. \quad (31)$$

(In the remainder of this paper, $\underline{\mathbf{T}}$ and $\hat{\underline{\mathbf{T}}}$ represent the same force state, but the latter denotes a function of $\underline{\mathbf{Y}}$, while the former denotes particular values of this function.) As shown in [2], peridynamic elastic materials have many of the same properties as elastic materials in the classical theory, including the reversible storage of energy supplied by external loads.

A key consideration in the process of shrinking the horizon to zero is that the bulk properties of the material should be unchanged during this process. To ensure this, a family of strain energy density functions parameterized by the horizon will be defined such that all these functions have the same response under homogeneous deformation.

Let an elastic material model be given with horizon δ_0 and strain energy density function $\hat{W}$; thus the force state is provided by (31). This *reference horizon* δ_0 will be held fixed throughout the remaining discussion. Consider a family of peridynamic elastic materials parameterized by variable horizon δ, and define

$$s = \delta/\delta_0,$$

so the shrinkage process means taking the limit as $s \to 0$. Let the strain energy density functions in this family of materials be given by

$$\hat{W}^s(\underline{\mathbf{Y}}) = \hat{W}(\underline{\mathbf{E}}^s(\underline{\mathbf{Y}})) \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V} \tag{32}$$

where $\underline{\mathbf{E}}^s(\underline{\mathbf{Y}})$ is the *enlarged deformation state* defined by

$$\underline{\mathbf{E}}^s(\underline{\mathbf{Y}})\langle\boldsymbol{\xi}\rangle = s^{-1}\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle \qquad \forall \boldsymbol{\xi} \in \mathcal{H}, \quad \forall \underline{\mathbf{Y}} \in \mathcal{V} \tag{33}$$

(Figure 1). To help provide a geometrical interpretation of $\underline{\mathbf{E}}^s(\underline{\mathbf{Y}})$, consider its value if the motion happens to be homogeneous with deformation gradient tensor $\mathbf{F}$. In this case, by (4), $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle = \mathbf{F}\boldsymbol{\xi}$; hence from (33),

$$\underline{\mathbf{E}}^s(\underline{\mathbf{Y}})\langle\boldsymbol{\xi}\rangle = s^{-1}\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle = s^{-1}\mathbf{F}s\boldsymbol{\xi} = \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle.$$

Thus, for any $s < 1$, $\underline{\mathbf{E}}^s(\underline{\mathbf{Y}})$ provides a view of the motion "through a microscope" such that stretches of short bonds are unchanged in the enlarged view.

To derive $\hat{\underline{\mathbf{T}}}^s$ from (32), consider a differential increment $d\underline{\mathbf{Y}}$ in the deformation state, and apply (31) and the defining relation of the Frechet derivative (25):

$$dW^s = \hat{\underline{\mathbf{T}}}^s(\underline{\mathbf{Y}}) \bullet d\underline{\mathbf{Y}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{E}}^s(\underline{\mathbf{Y}})) \bullet d\underline{\mathbf{E}}^s(\underline{\mathbf{Y}}).$$

From this, (33), and the definition of the dot product for vector states (23),

$$\int_{\mathcal{H}^s} \underline{\mathbf{T}}^s\langle\boldsymbol{\zeta}\rangle \cdot d\underline{\mathbf{Y}}\langle\boldsymbol{\zeta}\rangle \; dV_{\boldsymbol{\zeta}} = \int_{\mathcal{H}} \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \cdot s^{-1} d\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}}$$

where $\mathcal{H}^s$ is a sphere of radius $\delta = s\delta_0$, $\underline{\mathbf{T}}^s = \hat{\underline{\mathbf{T}}}^s(\underline{\mathbf{Y}})$, and $\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{E}}^s(\underline{\mathbf{Y}}))$. Changing the dummy variable of integration on the left side from $\boldsymbol{\zeta}$ to $s\boldsymbol{\xi}$ results in

$$\int_{\mathcal{H}} \underline{\mathbf{T}}^s\langle s\boldsymbol{\xi}\rangle \cdot d\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle \; (s^3 dV_{\boldsymbol{\xi}}) = \int_{\mathcal{H}} \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \cdot s^{-1} d\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}},$$

hence

$$\int_{\mathcal{H}} \left(\underline{\mathbf{T}}^s\langle s\boldsymbol{\xi}\rangle - s^{-4}\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle\right) \cdot d\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}} = 0. \tag{34}$$

Since (34) holds for any $d\underline{\mathbf{Y}}$, it follows that

$$\hat{\underline{\mathbf{T}}}^s(\underline{\mathbf{Y}})\langle s\boldsymbol{\xi}\rangle = s^{-4}\hat{\underline{\mathbf{T}}}(\underline{\mathbf{E}}^s(\underline{\mathbf{Y}}))\langle\boldsymbol{\xi}\rangle \qquad \forall \boldsymbol{\xi} \in \mathcal{H}, \quad \forall \underline{\mathbf{Y}} \in \mathcal{V}. \tag{35}$$

To account for nonhomogeneity of a body, $\mathbf{x}$ will now be included explicitly in the constitutive model: $\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x})$.

8 Convergence of the peridynamic stress field

A given motion $\mathbf{y}$ is assumed, independent of s. The following assumptions will be made that permit a meaningful comparison between the classical and peridynamic models:

(i) The motion $\mathbf{y}$ is a twice continuously differentiable function of $\mathbf{x}$ and t.

(ii) The constitutive model $\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x})$ is a continuously differentiable function of $\underline{\mathbf{Y}}$ and $\mathbf{x}$.

Let $\mathbf{F}$ denote the usual deformation gradient tensor field,

$$\mathbf{F}(\mathbf{x}, t) = \frac{\partial \mathbf{y}}{\partial \mathbf{x}}(\mathbf{x}, t) \qquad \forall \mathbf{x} \in \mathcal{B}, \quad t \geq 0. \tag{36}$$

In the following discussion, the time variable t will be omitted to make the notation more concise. Assumption (i) immediately implies

$$\underline{\mathbf{Y}}[\mathbf{x}]\langle \boldsymbol{\xi} \rangle = \mathbf{F}(\mathbf{x})\boldsymbol{\xi} + O(|\boldsymbol{\xi}|^2) \qquad \forall \boldsymbol{\xi} \in \mathcal{H}. \tag{37}$$

Consider the behavior of $\underline{\mathbf{Y}}$ near some $\mathbf{x}$. For any increment $\Delta\mathbf{x}$, define $\Delta\underline{\mathbf{Y}}$ by

$$\Delta\underline{\mathbf{Y}} = \underline{\mathbf{Y}}[\mathbf{x} + \Delta\mathbf{x}] - \underline{\mathbf{Y}}[\mathbf{x}]. \tag{38}$$

Using assumption (i), (4) and (38) imply

$$\Delta\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle = \mathbf{y}(\mathbf{x} + \Delta\mathbf{x} + \boldsymbol{\xi}) - \mathbf{y}(\mathbf{x} + \Delta\mathbf{x}) - \mathbf{y}(\mathbf{x} + \boldsymbol{\xi}) + \mathbf{y}(\boldsymbol{\xi}). \tag{39}$$

Suppose both of the following hold:

$$\Delta\mathbf{x} = O(s), \qquad \boldsymbol{\xi} = O(s). \tag{40}$$

Expanding each of the $\mathbf{y}$ terms in (39) as a Taylor series with remainder, using (36) and (40), yields

$$\Delta\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle = (\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}))\Delta\mathbf{x}\boldsymbol{\xi} + O(s^3) \quad \text{or} \quad \Delta\underline{Y}_i\langle \boldsymbol{\xi} \rangle = F_{ij,k}(\mathbf{x})\Delta x_j \xi_k + O(s^3). \tag{41}$$

The notation $F_{ij,k} = \partial F_{ij}/\partial x_k$ is used in (41).

Now consider the limiting behavior of the peridynamic stress tensor $\boldsymbol{\nu}^s(\mathbf{x})$ as s becomes small. From (30),

$$\boldsymbol{\nu}^s(\mathbf{x}) = \int_{\mathcal{S}} \int_0^{s\delta_0} \int_0^{s\delta_0} (y+z)^2 \, \hat{\underline{\mathbf{T}}}^s(\underline{\mathbf{Y}}[\mathbf{x}-z\mathbf{m}], \mathbf{x}-z\mathbf{m})\langle (y+z)\mathbf{m} \rangle \otimes \mathbf{m} \; dz \; dy \; d\Omega_{\mathbf{m}}. \tag{42}$$

Using (35) and the change of dummy variables $y \to sy$ and $z \to sz$,

$$\begin{aligned}
\boldsymbol{\nu}^s(\mathbf{x}) &= \int_{\mathcal{S}} \int_0^{\delta_0} \int_0^{\delta_0} (sy+sz)^2 \left(s^{-4} \underline{\hat{\mathbf{T}}}(\underline{\mathbf{E}}^s(\underline{\mathbf{Y}}[\mathbf{x} - sz\mathbf{m}]), \mathbf{x} - sz\mathbf{m}) \langle (y+z)\mathbf{m} \rangle \right) \\
&\quad \otimes \mathbf{m} \; (sdz) \; (sdy) \; d\Omega_{\mathbf{m}} \\
&= \int_{\mathcal{S}} \int_0^{\delta_0} \int_0^{\delta_0} (y+z)^2 \underline{\hat{\mathbf{T}}}(\underline{\mathbf{E}}^s(\underline{\mathbf{Y}}[\mathbf{x} - sz\mathbf{m}]), \mathbf{x} - sz\mathbf{m}) \langle (y+z)\mathbf{m} \rangle \\
&\quad \otimes \mathbf{m} \; dz \; dy \; d\Omega_{\mathbf{m}}. \qquad (43)
\end{aligned}$$

Now observe that by assumption (i) and equations (33), (37), (38), and (41) with $\Delta\mathbf{x} = -sz\mathbf{m}$, we have that for any $\boldsymbol{\xi} \in \mathcal{H}$,

$$\begin{aligned}
\underline{\mathbf{E}}^s(\underline{\mathbf{Y}}[\mathbf{x} - sz\mathbf{m}])\langle \boldsymbol{\xi} \rangle &= s^{-1} \underline{\mathbf{Y}}[\mathbf{x} - sz\mathbf{m}] \langle s\boldsymbol{\xi} \rangle \\
&= s^{-1} \left(\underline{\mathbf{Y}}[\mathbf{x}]\langle s\boldsymbol{\xi} \rangle + \Delta\underline{\mathbf{Y}}\langle s\boldsymbol{\xi} \rangle \right) \\
&= s^{-1} \left(\underline{\mathbf{Y}}[\mathbf{x}]\langle s\boldsymbol{\xi} \rangle + (\nabla_{\mathbf{x}} \mathbf{F}(\mathbf{x}))(-sz\mathbf{m})(s\boldsymbol{\xi}) + O(s^3) \right) \\
&= s^{-1} \left(\underline{\mathbf{Y}}[\mathbf{x}]\langle s\boldsymbol{\xi} \rangle + O(s^2) \right) \\
&= s^{-1} \left(\mathbf{F}(\mathbf{x}) s\boldsymbol{\xi} + O(s^2) \right) \\
&= \mathbf{F}(\mathbf{x})\boldsymbol{\xi} + O(s). \qquad (44)
\end{aligned}$$

From (44),

$$\underline{\mathbf{E}}^s(\underline{\mathbf{Y}}[\mathbf{x} - sz\mathbf{m}]) = \mathbf{F}(\mathbf{x})\underline{\mathbf{X}} + O(s) \qquad (45)$$

where $\underline{\mathbf{X}}$ is the identity vector state defined in (24). To further simplify the integrand in (43), use (45) and assumption (ii) to yield

$$\begin{aligned}
&\underline{\hat{\mathbf{T}}}(\underline{\mathbf{E}}^s(\underline{\mathbf{Y}}[\mathbf{x} - sz\mathbf{m}]), \mathbf{x} - sz\mathbf{m}) \langle (y+z)\mathbf{m} \rangle \\
&\qquad = \underline{\hat{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}} + O(s), \mathbf{x} - sz\mathbf{m}) \langle (y+z)\mathbf{m} \rangle \\
&\qquad = \underline{\hat{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}}, \mathbf{x}) \langle (y+z)\mathbf{m} \rangle + O(s). \qquad (46)
\end{aligned}$$

From (43) and (46),

$$\begin{aligned}
\boldsymbol{\nu}^s(\mathbf{x}) &= \int_{\mathcal{S}} \int_0^{\delta_0} \int_0^{\delta_0} (y+z)^2 \underline{\hat{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}}, \mathbf{x}) \langle (y+z)\mathbf{m} \rangle \otimes \mathbf{m} \; dz \; dy \; d\Omega_{\mathbf{m}} + O(s) \\
&= \int_{\mathcal{S}} \int_0^{\delta_0} \int_z^{\delta_0} p^2 \underline{\hat{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}}, \mathbf{x}) \langle p\mathbf{m} \rangle \otimes \mathbf{m} \; dp \; dz \; d\Omega_{\mathbf{m}} + O(s) \qquad (47)
\end{aligned}$$

where the change of variables $p = y + z$ has been used. The upper limit of integration on the integral over p is shown as δ_0 instead of $\delta_0 + z$ because of (3).

Using Lemma 1 (see Appendix) in (47) with $g(p) = p^2\underline{\hat{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}}, \mathbf{x})\langle p\mathbf{m}\rangle$, it follows that

$$\begin{aligned}\boldsymbol{\nu}^s(\mathbf{x}) &= \int_{\mathcal{S}} \int_0^{\delta_0} p^3 \underline{\hat{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}}, \mathbf{x})\langle p\mathbf{m}\rangle \otimes \mathbf{m}\; dp\; d\Omega_{\mathbf{m}} + O(s) \\ &= \int_{\mathcal{H}} |\mathbf{p}|\; \underline{\hat{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}}, \mathbf{x})\langle \mathbf{p}\rangle \otimes \mathbf{m}\; dV_{\mathbf{p}} + O(s) \\ &= \int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}}, \mathbf{x})\langle \mathbf{p}\rangle \otimes \mathbf{p}\; dV_{\mathbf{p}} + O(s) \qquad (48)\end{aligned}$$

in which the change of variables $\mathbf{p} = p\mathbf{m}$ was used, hence $dV_{\mathbf{p}} = p^2 dp d\Omega_{\mathbf{m}}$. Now define the *collapsed peridynamic stress tensor field* $\boldsymbol{\nu}^0$ by

$$\boldsymbol{\nu}^0(\mathbf{x}) = \int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}}, \mathbf{x})\langle \boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\; dV_{\boldsymbol{\xi}} \qquad \forall \mathbf{x} \in \mathcal{B}. \qquad (49)$$

Geometrically, $\mathbf{F}(\mathbf{x})\underline{\mathbf{X}}$ represents the deformation state $\underline{\mathbf{Y}}$ that would be obtained by observing the motion at $\mathbf{x}$ "through a microscope" as suggested by Figure 1. The discussion above has established the following proposition.

Proposition 1. Let $\mathcal{B}$ be an open region occupied by the reference configuration of an elastic peridynamic body, and let $\mathbf{y}$ be a motion of $\mathcal{B}$. Let $\hat{W}$ be a strain energy density function for the body with horizon δ_0, and let $\underline{\hat{\mathbf{T}}}$ be the corresponding constitutive model derived from (31). Suppose that assumptions (i) and (ii) are satisfied. Let a family of constitutive models parameterized by horizon $\delta = s\delta_0$ be given by (32) for any $s > 0$. Let $\boldsymbol{\nu}^s$ be the corresponding family of peridynamic stress tensor fields defined by (42). Then

$$\lim_{s\to 0} \boldsymbol{\nu}^s = \boldsymbol{\nu}^0 \qquad \text{on } \mathcal{B}$$

where $\boldsymbol{\nu}^0$ is the tensor field defined by (49).

The condition stated in Proposition 1 that $\mathcal{B}$ is an open set is required so that for sufficiently small s, the neighborhood of radius s centered at any $\mathbf{x} \in \mathcal{B}$ is contained in $\mathcal{B}$. This is needed for statements such as (37) to be true. Proposition 1 still holds if assumption (i) is replaced by the weaker assumption that $\mathbf{y}$ is a continuously differentiable function of $\mathbf{x}$. However, the stronger assumptions will be needed for subsequent results below. The following proposition follows immediately from (49) and assumptions (i) and (ii):

Proposition 2. Under the conditions of Proposition 1, $\boldsymbol{\nu}^0$ is a continuously differentiable function of $\mathbf{x}$.

At this point, we briefly return to the question raised in Section 3 of whether the classical model is accurate when separated forces are present. Although this is a complicated question, one way to approach it is recast it in the following form. Suppose we have a local material model $\hat{\boldsymbol{\nu}}^0$ that has been calibrated according to bulk material properties based on laboratory data obtained for large specimens. Suppose we want to know how the presence of separated forces would affect the stress field in a certain motion. Let the interaction distance for these separated forces be $s\delta_0$ (arbitrarily set $\delta_0 = 1$), and let the corresponding peridynamic material model, which includes the separated forces, result in a peridynamic stress field $\boldsymbol{\nu}^s$.

Now we can ask how changes in s affect $\boldsymbol{\nu}^s$ for this particular motion. To answer this, expand out the $O(s)$ error term in (44), then use this in (46) and (43) to obtain

$$|\boldsymbol{\nu}^s(\mathbf{x}) - \boldsymbol{\nu}^0(\mathbf{x})| \sim |\nabla \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}})||\nabla_{\mathbf{x}} \mathbf{F}(\mathbf{x})|s \qquad \text{as } s \to 0. \tag{50}$$

The conclusion is that for a given bulk material response, separated forces result in changes to the stress field on the order of the second derivatives of displacement. Because $\nabla_{\mathbf{x}}\mathbf{F}$ appears in (50), this result is suggestive of the strain gradient approach to nonlocality [24, 25, 26]. However, the peridynamic stress tensor would be expected to differ significantly from a stress tensor based on a strain gradient theory in the vicinity of a strong discontinuity. It is perhaps noteworthy that the size of the body as a whole does not appear in (50). Although this overall size might be one factor affecting $\nabla_{\mathbf{x}}\mathbf{F}$, this gradient might also be more strongly affected by conditions within the motion, such as localization, cracking, and the presence of interfaces. This observation would tend to support the view expressed in Section 3 that the classical theory is not necessarily accurate for large bodies.

9 Convergence of the divergence of the peridynamic stress field

Propositions 1 and 2 do not by themselves establish that the integral in the equation of motion (1) converges to $\nabla \cdot \boldsymbol{\nu}^0$ as $s \to 0$. However, this convergence will now be shown directly. Let a motion $\mathbf{y}$ on $\mathcal{B}$ be given. Define the following functional of $\mathbf{u}$, parameterized by s:

$$\mathbf{L}^s_{\mathbf{u}}(\mathbf{x}) = \int_{\mathcal{H}^s} \{\underline{\mathbf{T}}^s[\mathbf{x}]\langle\boldsymbol{\zeta}\rangle - \underline{\mathbf{T}}^s[\mathbf{x}+\boldsymbol{\zeta}]\langle -\boldsymbol{\zeta}\rangle\} \ dV_{\boldsymbol{\zeta}} \quad \forall \mathbf{x} \in \mathcal{B}. \tag{51}$$

(Time labels will be omitted to simplify the notation, although it is understood that $\mathbf{u}$ can depend on time.) From (35) and (51), and setting $\boldsymbol{\zeta} = s\boldsymbol{\xi}$,

$$\begin{aligned}\mathbf{L}^s_{\mathbf{u}}(\mathbf{x}) &= \int_{\mathcal{H}} \left\{ s^{-4}\underline{\mathbf{T}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle - s^{-4}\underline{\mathbf{T}}[\mathbf{x}+s\boldsymbol{\xi}]\langle -\boldsymbol{\xi}\rangle \right\}\ s^3 dV_{\boldsymbol{\xi}} \\ &= s^{-1}\int_{\mathcal{H}} \{\underline{\mathbf{T}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle - \underline{\mathbf{T}}[\mathbf{x}+s\boldsymbol{\xi}]\langle -\boldsymbol{\xi}\rangle\}\ dV_{\boldsymbol{\xi}} \qquad (52)\end{aligned}$$

where

$$\underline{\mathbf{T}}[\mathbf{x}] = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{E}}^s(\underline{\mathbf{Y}}[\mathbf{x}]), \mathbf{x}), \qquad \underline{\mathbf{T}}[\mathbf{x}+s\boldsymbol{\xi}] = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{E}}^s(\underline{\mathbf{Y}}[\mathbf{x}+s\boldsymbol{\xi}]), \mathbf{x}+s\boldsymbol{\xi}).$$

Setting

$$\Delta\underline{\mathbf{Y}} = \underline{\mathbf{Y}}[\mathbf{x}+s\boldsymbol{\xi}] - \underline{\mathbf{Y}}[\mathbf{x}], \qquad \Delta\underline{\mathbf{E}}^s = \underline{\mathbf{E}}^s(\underline{\mathbf{Y}}[\mathbf{x}+s\boldsymbol{\xi}]) - \underline{\mathbf{E}}^s(\underline{\mathbf{Y}}[\mathbf{x}]),$$

it follows from (33) that

$$\Delta\underline{\mathbf{E}}^s = \underline{\mathbf{E}}^s(\Delta\underline{\mathbf{Y}}). \qquad (53)$$

Applying (33) again to write out $\Delta\underline{\mathbf{E}}^s$ explicitly, and using (41) to approximate the result for small s, for any $\boldsymbol{\xi}, \boldsymbol{\xi}' \in \mathcal{H}$,

$$\begin{aligned}\Delta\underline{\mathbf{E}}^s\langle\boldsymbol{\xi}'\rangle &= s^{-1}\Delta\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}'\rangle \\ &= s^{-1}\left((\nabla_{\mathbf{x}}\mathbf{F})(s\boldsymbol{\xi}')(s\boldsymbol{\xi}) + O(s^3)\right) \\ &= s(\nabla_{\mathbf{x}}\mathbf{F})\boldsymbol{\xi}'\boldsymbol{\xi} + O(s^2)\end{aligned}$$

or

$$\Delta\underline{\mathbf{E}}^s = s(\nabla_{\mathbf{x}}\mathbf{F}\underline{\mathbf{X}})\boldsymbol{\xi} + O(s^2). \qquad (54)$$

(Recall from (24) that $\mathbf{F}\underline{\mathbf{X}}\langle\boldsymbol{\xi}'\rangle = \mathbf{F}\boldsymbol{\xi}'$. The identity vector state $\underline{\mathbf{X}}$ does not depend on $\mathbf{x}$.) To evaluate the second term in the integrand in (52), use assumptions (i) and (ii), the first two terms of a Taylor series with remainder, and (54) to obtain

$$\begin{aligned}\underline{\mathbf{T}}[\mathbf{x}+s\boldsymbol{\xi}] &= \underline{\mathbf{T}}[\mathbf{x}] + \nabla\hat{\underline{\mathbf{T}}}\bullet\Delta\underline{\mathbf{E}}^s + (\nabla_{\mathbf{x}}\hat{\underline{\mathbf{T}}})s\boldsymbol{\xi} + O(s^2) \\ &= \underline{\mathbf{T}}[\mathbf{x}] + s(\nabla\hat{\underline{\mathbf{T}}}\bullet(\nabla_{\mathbf{x}}\mathbf{F}\underline{\mathbf{X}}) + \nabla_{\mathbf{x}}\hat{\underline{\mathbf{T}}})\boldsymbol{\xi} + O(s^2). \qquad (55)\end{aligned}$$

The term $\nabla_{\mathbf{x}}\hat{\underline{\mathbf{T}}}$ refers to the explicit dependence of $\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x})$ on $\mathbf{x}$ due to nonhomogeneity. Using (55) in (52) with the change of dummy variable $\boldsymbol{\xi} \to -\boldsymbol{\xi}$, applying the chain rule, and noting that the zero-order terms $\underline{\mathbf{T}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle$ and $\underline{\mathbf{T}}[\mathbf{x}]\langle -\boldsymbol{\xi}\rangle$ cancel each other when the integration is carried out,

$$\begin{aligned}\mathbf{L}^s_{\mathbf{u}}(\mathbf{x}) &= \int_{\mathcal{H}} (\nabla\hat{\underline{\mathbf{T}}}\bullet(\nabla_{\mathbf{x}}\mathbf{F}\underline{\mathbf{X}}) + \nabla_{\mathbf{x}}\hat{\underline{\mathbf{T}}})\langle\boldsymbol{\xi}\rangle\boldsymbol{\xi}\ dV_{\boldsymbol{\xi}} + O(s) \\ &= \nabla\cdot\int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle\otimes\boldsymbol{\xi}\ dV_{\boldsymbol{\xi}} + O(s) \\ &= \nabla\cdot\boldsymbol{\nu}^0(\mathbf{x}) + O(s) \qquad (56)\end{aligned}$$

where $\nabla\cdot$ denotes the divergence operator and the last step comes from (49). This proves the following result.

Proposition 3. Under the conditions of Proposition 1,

$$\lim_{s\to 0} \mathbf{L}^s_{\mathbf{u}} = \nabla \cdot \boldsymbol{\nu}^0 \qquad \text{on } \mathcal{B}$$

where $\mathbf{L}^s_{\mathbf{u}}$ is defined by (51).

10 Constitutive model for the collapsed peridynamic stress tensor

Since (49) provides an expression for the collapsed peridynamic stress tensor at $\mathbf{x}$ that depends only on $\mathbf{F}(\mathbf{x})$, we can now define a constitutive model for this $\boldsymbol{\nu}^0$ as follows:

$$\hat{\boldsymbol{\nu}}^0(\mathbf{F},\mathbf{x}) = \int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}},\mathbf{x})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\; dV_{\boldsymbol{\xi}} \qquad \forall\mathbf{F}\in\mathcal{L},\ \forall\mathbf{x}\in\mathcal{B} \tag{57}$$

where $\mathcal{L}$ is the set of all second order tensors. Recall that $\mathbf{F}\underline{\mathbf{X}}$ is a vector state; see (22) and (24) regarding notation.

Equation (57) is a *local* constitutive model in the sense that it depends on the motion only through the deformation gradient tensor. (It can also depend on $\mathbf{x}$ explicitly to reflect nonhomogeneity of the body.) As shown by Proposition 3 and (1), the $\boldsymbol{\nu}^0$ field provided by this constitutive model describes the internal forces to which the peridynamic model converges (subject to assumptions (i) and (ii)) in the limit of small horizon. In the remainder of this section we consider the properties of $\hat{\boldsymbol{\nu}}^0$, in the sense of the classical theory, with regard to angular momentum balance, isotropy, and objectivity. The function $\hat{\boldsymbol{\nu}}^0$ will be referred to as a "stress-strain relation" to distinguish it from a peridynamic constitutive model and to reflect its dependence on the strain-like quantity $\mathbf{F}$.

10.1 Angular momentum balance

To complete the identification of $\hat{\boldsymbol{\nu}}^0$ defined by (57) with the Piola-Kirchhoff stress in the classical theory, we now investigate the properties of the corresponding Cauchy stress defined by

$$\hat{\boldsymbol{\tau}}^0 = \frac{1}{J}\hat{\boldsymbol{\nu}}^0\mathbf{F}^T, \qquad J = \det\mathbf{F} \qquad \forall\mathbf{F}\in\mathcal{L} \tag{58}$$

where it is assumed that $J \neq 0$ [27]. As described in [2], a sufficient condition for global balance of angular momentum to hold in a peridynamic body is that the constitutive model obey

$$\int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}} = \mathbf{0} \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V}, \; \forall \mathbf{x} \in \mathcal{B} \tag{59}$$

or using components,

$$\varepsilon_{ijk} \int_{\mathcal{H}} \underline{\hat{T}}_j(\underline{\mathbf{Y}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \underline{Y}_k\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}} = 0 \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V}, \; \forall \mathbf{x} \in \mathcal{B} \tag{60}$$

where ε_{ijk} is the alternator symbol. Geometrically, the condition (59) means that force states individually satisfy balance of angular momentum; *i.e.,* the forces on $\mathbf{x}$ due to $\underline{\mathbf{T}}[\mathbf{x}]$ exert no net moment.

Proposition 4. Under the conditions of Proposition 1, suppose $\underline{\hat{\mathbf{T}}}$ satisfies (59). For any $\mathbf{x} \in \mathcal{B}$, let $\hat{\boldsymbol{\nu}}^0$ be given by (57), and let $\hat{\boldsymbol{\tau}}^0$ be given by (58). Then $\hat{\boldsymbol{\tau}}^0$ is symmetric on $\mathcal{B}$.

Proof. Setting $\underline{\mathbf{Y}} = \mathbf{F}\underline{\mathbf{X}}$ in (60) and using (57) leads to

$$\begin{aligned}
0 &= \varepsilon_{ijk} \int_{\mathcal{H}} \underline{\hat{T}}_j(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle (\mathbf{F}\underline{\mathbf{X}})_{km}\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}} \\
&= \varepsilon_{ijk} \int_{\mathcal{H}} \underline{\hat{T}}_j(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle F_{km}\xi_m \, dV_{\boldsymbol{\xi}} \\
&= \varepsilon_{ijk} \left(\int_{\mathcal{H}} \underline{\hat{T}}_j(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \xi_m \, dV_{\boldsymbol{\xi}} \right) F_{km} \\
&= \varepsilon_{ijk} \hat{\nu}^0_{jm} F_{km} \\
&= J \varepsilon_{ijk} \hat{\tau}^0_{jk}
\end{aligned}$$

so $\hat{\tau}^0_{jk} = \hat{\tau}^0_{kj}$. $\square$

10.2 Isotropy

If $\mathbf{Q}$ is any orthogonal tensor, then the corresponding orthogonal state $\underline{\mathbf{Q}}$ is defined by

$$\underline{\mathbf{Q}}\langle\boldsymbol{\xi}\rangle = \mathbf{Q}\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathcal{H}.$$

As discussed in [2], the condition for isotropy in a peridynamic body is

$$\underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}} \circ \underline{\mathbf{Q}}, \mathbf{x}) = \underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x}) \circ \underline{\mathbf{Q}} \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V}, \; \forall \mathbf{x} \in \mathcal{B} \tag{61}$$

for all orthogonal states $\underline{\mathbf{Q}}$.

Proposition 5. Under the conditions of Proposition 1, suppose $\underline{\hat{\mathbf{T}}}$ satisfies (61). For any $\mathbf{x} \in \mathcal{B}$, let $\hat{\boldsymbol{\nu}}^0$ be given by (57). Then

$$\hat{\boldsymbol{\nu}}^0(\mathbf{FQ}, \mathbf{x}) = \hat{\boldsymbol{\nu}}^0(\mathbf{F}, \mathbf{x})\mathbf{Q} \tag{62}$$

for all orthogonal tensors $\mathbf{Q}$ and all $\mathbf{F}$.

Proof. For any orthogonal tensor $\mathbf{Q}$ and any $\mathbf{F}$, using (57) and (61) and the change of variables $\boldsymbol{\xi}' = \mathbf{Q}\boldsymbol{\xi}$,

$$\begin{aligned}
\hat{\boldsymbol{\nu}}^0(\mathbf{FQ}, \mathbf{x}) &= \int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\mathbf{FQ}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\; dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}} \circ \underline{\mathbf{Q}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\; dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} (\underline{\hat{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x}) \circ \underline{\mathbf{Q}})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\; dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\mathbf{Q}\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\; dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}'\rangle \otimes (\mathbf{Q}^T\boldsymbol{\xi}')\; dV_{\boldsymbol{\xi}'} \\
&= \left(\int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}'\rangle \otimes \boldsymbol{\xi}'\; dV_{\boldsymbol{\xi}'}\right)\mathbf{Q} \\
&= \hat{\boldsymbol{\nu}}^0(\mathbf{F}, \mathbf{x})\mathbf{Q}. \qquad \square
\end{aligned}$$

Equation (62) is the condition for isotropy of a (local) material model in the classical theory in terms of the Piola-Kirchhoff stress [27]. So, the conclusion is that if the peridynamic material model is isotropic, then the corresponding $\hat{\boldsymbol{\nu}}^0$ is also isotropic in the sense of the classical theory.

10.3 Objectivity

As discussed in [2], the condition for objectivity in a peridynamic body is

$$\underline{\hat{\mathbf{T}}}(\underline{\mathbf{Q}} \circ \underline{\mathbf{Y}}, \mathbf{x}) = \underline{\mathbf{Q}} \circ \underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x}) \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V},\ \forall \mathbf{x} \in \mathcal{B}. \tag{63}$$

for all orthogonal states $\underline{\mathbf{Q}}$.

Proposition 6. Under the conditions of Proposition 1, suppose $\underline{\hat{\mathbf{T}}}$ satisfies (63). For any $\mathbf{x} \in \mathcal{B}$, let $\hat{\boldsymbol{\nu}}^0$ be given by (57). Then

$$\hat{\boldsymbol{\nu}}^0(\mathbf{QF}, \mathbf{x}) = \mathbf{Q}\hat{\boldsymbol{\nu}}^0(\mathbf{F}, \mathbf{x}) \tag{64}$$

for all orthogonal tensors $\mathbf{Q}$ and all $\mathbf{F}$.

Proof. For any orthogonal tensor $\mathbf{Q}$ and any $\mathbf{F}$, from (57) and (63),

$$\begin{aligned}
\hat{\boldsymbol{\nu}}^0(\mathbf{QF}, \mathbf{x}) &= \int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\mathbf{QF}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Q}} \circ \mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} (\underline{\mathbf{Q}} \circ \hat{\underline{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x}))\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} \mathbf{Q}\hat{\underline{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\xi}} \\
&= \mathbf{Q}\hat{\boldsymbol{\nu}}^0(\mathbf{F}, \mathbf{x}). \qquad \square
\end{aligned}$$

(64) is the condition for a classical constitutive model to be objective expressed in terms of the Piola-Kirchhoff stress [27]. From this result, the conclusion is that if a peridynamic constitutive model is objective, then the corresponding $\hat{\boldsymbol{\nu}}^0$ is also objective in the sense of the classical theory.

10.4 Hyperelasticity

In this section it will be shown that $\hat{\boldsymbol{\nu}}^0$ is derivable from a scalar valued strain energy density function via the usual tensor gradient within the classical theory. To do this, define the *collapsed strain energy density function* $\hat{W}^0 : \mathcal{L} \times \mathcal{B} \to \mathbb{R}$ by

$$\hat{W}^0(\mathbf{F}, \mathbf{x}) = \hat{W}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x}) \qquad \forall \mathbf{F} \in \mathcal{L},\ \forall \mathbf{x} \in \mathcal{B}. \tag{65}$$

where $\hat{W}$ is the peridynamic strain energy density function in (31). Denote the tensor gradient by $\partial/\partial\mathbf{F}$, thus

$$\left(\frac{\partial \hat{W}^0}{\partial \mathbf{F}}\right)_{ij} = \frac{\partial \hat{W}^0}{\partial F_{ij}}. \tag{66}$$

Proposition 7. Let a peridynamic elastic strain energy density function $\hat{W} : \mathcal{V} \times \mathcal{B} \to \mathbb{R}$ for a nonhomogeneous body $\mathcal{B}$ be given, and let $\underline{\mathbf{T}} = \nabla \hat{W}$, where ∇ denotes the Frechet derivative. Let $\hat{\boldsymbol{\nu}}^0$ be given by (57). Let $\hat{W}^0$ be defined by (65). Then

$$\hat{\boldsymbol{\nu}}^0(\mathbf{F}, \mathbf{x}) = \frac{\partial \hat{W}^0}{\partial \mathbf{F}}(\mathbf{F}, \mathbf{x}) \qquad \forall \mathbf{F} \in \mathcal{L},\ \forall \mathbf{x} \in \mathcal{B}. \tag{67}$$

Proof. Since

$$(\mathbf{F}\underline{\mathbf{X}})_k\langle\boldsymbol{\xi}\rangle = F_{km}\xi_m,$$

it follows that

$$\left(\frac{\partial(\mathbf{F}\underline{\mathbf{X}})}{\partial \mathbf{F}}\langle\boldsymbol{\xi}\rangle\right)_{kij} = \frac{\partial}{\partial F_{ij}}(F_{km}\xi_m) = \delta_{ik}\delta_{jm}\xi_m = \delta_{ik}\xi_j. \tag{68}$$

By definition, $\hat{W}$ is differentiable (in the sense of Frechet derivatives). From (65) and the chain rule,

$$\frac{\partial \hat{W}^0}{\partial \mathbf{F}} = \nabla\hat{W} \bullet \frac{\partial(\mathbf{F}\underline{\mathbf{X}})}{\partial \mathbf{F}}.$$

Expressing this in component form, expanding out the dot product, and using (68),

$$\begin{aligned}\left(\frac{\partial \hat{W}^0}{\partial \mathbf{F}}\right)_{ij} &= \int_{\mathcal{H}} \underline{T}_k\langle\boldsymbol{\xi}\rangle \left(\frac{\partial(\mathbf{F}\underline{\mathbf{X}})}{\partial \mathbf{F}}\langle\boldsymbol{\xi}\rangle\right)_{kij} dV_{\boldsymbol{\xi}} \\ &= \int_{\mathcal{H}} \underline{T}_k\langle\boldsymbol{\xi}\rangle\delta_{ik}\xi_j \, dV_{\boldsymbol{\xi}} \\ &= \int_{\mathcal{H}} \underline{T}_i\langle\boldsymbol{\xi}\rangle\xi_j \, dV_{\boldsymbol{\xi}} \\ &= \hat{\nu}^0_{ij}\end{aligned}$$

where the final step comes from (57). □

One implication of Proposition 7 is that the collapsed peridynamic stress tensor $\hat{\boldsymbol{\nu}}^0$ is conjugate to $\mathbf{F}$, which is consistent with the properties of Piola-Kirchhoff stress tensors in the classical theory for hyperelastic materials.

11 Discussion

The above development has shown that under the assumptions (i) and (ii), the elastic peridynamic model converges to the classical model in the limit of small horizon. Starting with any peridynamic strain energy function $\hat{W}$, and defining a family of peridynamic materials by (32) for variable horizon while holding the bulk properties fixed, the limiting stress tensor is provided by (57). This is a local stress-strain relation. The stress tensor is obtainable from the tensor gradient of the strain energy density function defined by (65). The resulting stress field $\boldsymbol{\nu}^0$ satisfies the classical equation of motion,

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \nabla\cdot\boldsymbol{\nu}^0(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t). \tag{69}$$

As shown in Section 10, the stress-strain relation (57) satisfies the conditions on the Piola-Kirchhoff stress in the classical theory for angular momentum balance, isotropy, and objectivity, provided the underlying peridynamic model meets these conditions. The condition $\delta < \infty$ is not required for the present results to hold, since a different quantity could be used to characterize the changing length scale, such as the location of a maximum in the dependence of bond force as a function of bond length.

If the assumptions (i) and (ii) fail to be satisfied, *i.e.,* if either the motion fails to be twice continuously differentiable, or if the peridynamic constitutive model fails to be continuously differentiable, then the conclusions regarding convergence to a classical model in the limit of small horizon fail to hold. In this case, the peridynamic equations continue to be applicable at any positive horizon, but convergence properties in the limit of zero horizon are undetermined. An example is provided by a peridynamic body containing a crack. Since the peridynamic model does not use the spatial derivatives of the motion, it continues to be applicable even on the crack surface. However, because these derivatives are undefined on the crack surface, expressions such as (49) cannot be used since $\mathbf{F}$ does not exist there. Nevertheless, the standard form of the jump condition for momentum across a surface of discontinuity continues to hold, even though $\boldsymbol{\nu}^0$ is not defined on the surface. In equilibrium, the applicable jump condition [28] is

$$(\boldsymbol{\nu}^0_+ - \boldsymbol{\nu}^0_-)\mathbf{n} = \mathbf{0}$$

where $\mathbf{n}$ is a unit vector normal to the surface of discontinuity, and the subscripts $+$ and $-$ refer to conditions immediately on either side of it. In addition to being consistent with this jump condition, the peridynamic model has been proposed as a framework in which to investigate the structure and energy balance in interfaces such as phase boundaries [29].

The present paper has been concerned with the process of starting with a peridynamic material model and shrinking the length scale to obtain a local model. Suppose we wish to go in the other direction? That is, suppose we have an expression for a stress tensor in the classical theory and ask whether there is a peridynamic constitutive model that is consistent with this. This question has been investigated in [2]; equation (142) of this reference provides the following force state for a given stress tensor $\boldsymbol{\sigma}$:

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \underline{\omega}\langle\boldsymbol{\xi}\rangle\boldsymbol{\sigma}(\mathbf{F})\mathbf{K}^{-1}\boldsymbol{\xi} \qquad \forall\boldsymbol{\xi} \in \mathcal{H} \tag{70}$$

where

$$\mathbf{K} = \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\boldsymbol{\xi} \otimes \boldsymbol{\xi}\; dV_{\boldsymbol{\xi}},$$

and $\underline{\omega}$ is a scalar state that acts as a weighting function. The value of $\mathbf{F}$ to be used in $\boldsymbol{\sigma}(\mathbf{F})$ is provided by

$$\mathbf{F} = \left(\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\; dV_{\boldsymbol{\xi}}\right)\mathbf{K}^{-1}.$$

In addition to providing a practical way that a given classical stress-strain model can incorporated into the peridynamic framework, (70) allows us to define a peridynamic version of the kinetic stress. The kinetic stress $\tilde{\boldsymbol{\tau}}$ accounts for the net momentum flux due to small random velocity fluctuations $\mathbf{v}$ superposed on the continuum velocity field [18]:

$$\tilde{\boldsymbol{\tau}} = -\rho\overline{\mathbf{v}\otimes\mathbf{v}}$$

where the overline denotes a statistical average of a random variable. To incorporate the kinetic stress into the peridynamic model, first use (58) to define the corresponding Piola-Kirchhoff stress:

$$\tilde{\boldsymbol{\nu}} = J\tilde{\tau}\mathbf{F}^{-T} \qquad J = \det\mathbf{F};$$

then set $\boldsymbol{\sigma} = \tilde{\boldsymbol{\nu}}$ in (70). The result is

$$\underline{\tilde{\mathbf{T}}}\langle\boldsymbol{\xi}\rangle = J\underline{\omega}\langle\boldsymbol{\xi}\rangle\tilde{\boldsymbol{\tau}}\mathbf{F}^{-T}\mathbf{K}^{-1}\boldsymbol{\xi} \qquad \text{or} \qquad \underline{\tilde{\mathbf{T}}} = J\underline{\omega}\tilde{\boldsymbol{\tau}}\mathbf{F}^{-T}\mathbf{K}^{-1}\underline{\mathbf{X}}.$$

By substituting $\underline{\tilde{\mathbf{T}}}$ into (49), one easily finds that the collapsed stress tensor corresponding to this force state is

$$\boldsymbol{\nu}^0 = \tilde{\boldsymbol{\nu}}. \tag{71}$$

This confirms that the peridynamic version of the kinetic stress is consistent with the local version in the limit of small horizon. The method outlined here for incorporating the kinetic stress into the peridynamic model does not re-examine the physical and statistical basis for the conventional view that it essentially a local quantity, free of any length scale. However, it appears possible that an alternative approach to the treatment of random fluctuations might reveal some physical aspect of nonlocality in the kinetic stress, with a length scale supplied perhaps by the mean free path, whose importance in determining the shear viscosity of gases is widely accepted.

Modeling of physical phenomena governed by diffusion within the peridynamic framework, including heat transport, is an area of current research. Mathematical modeling of heat transport using local concepts that lead to the standard partial differential equations causes difficulties similar to those that occur in mechanics. For example, problems involving moving phase boundaries, such as the Stefan problem, lead to singularities that require special treatment within the standard theory of heat transport [30]. Other types of singularities in heat transfer arise in technologically important problems such as the rewetting problem in nuclear reactors. In the rewetting problem, nonmonotonicity in the relation between heat transport rate and temperature along the surface of an internally heated, submerged rod leads to the emergence of a moving singularity in the equations [31]. This type of singularity is analogous to a crack tip singularity in mechanics, because it can imply unbounded heat flux. It therefore appears possible that a nonlocal treatment of heat transport analogous to the peridynamic mechanical model could prove to be useful in these problems.

The classical theory of elasticity is generally regarded as not having a length scale. The results of the present paper suggest that a valid alternative view of the classical theory is that it does have a length scale, and the value of this length scale is zero.

Appendix

The following result is used in equation (48).

Lemma 1. Let a be a positive number, and let $g : [0, a] \to \mathbb{R}$ be an integrable function. Let

$$I(a) = \int_0^a \int_z^a g(p)\ dp\ dz. \tag{72}$$

Then

$$I(a) = \int_0^a pg(p)\ dp. \tag{73}$$

Proof. Define $k(z, a) = \int_z^a g(p)dp$; thus $I(a) = \int_0^a k(z, a)dz$. Differentiating this,

$$\frac{dI}{da} = k(a, a) + \int_0^a \frac{\partial k}{\partial a}(z, a)\ dz = 0 + \int_0^a g(a)\ dz = ag(a).$$

This is a first order differential equation whose solution under the boundary condition $I(0) = 0$, which is implied by (72), is given by (73). $\square$

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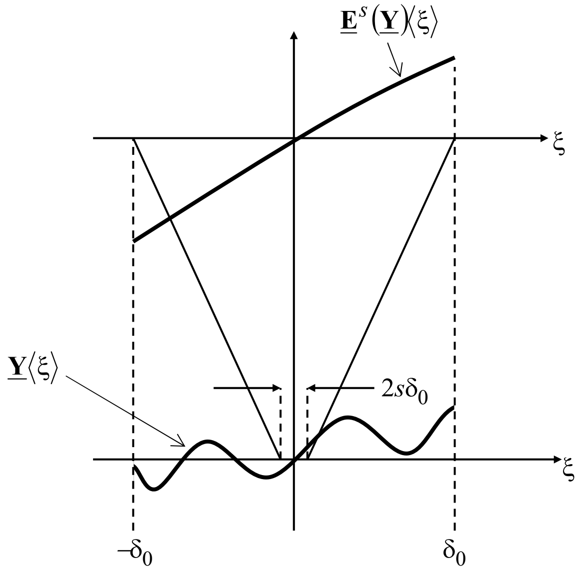

Figure 1: The enlarged deformation state $\underline{\mathbf{E}}^s(\underline{\mathbf{Y}})$ maps the part of the deformation state $\underline{\mathbf{Y}}$ within the small horizon $s\delta_0$ to the original horizon δ_0.